29/11/23

MATH 2050 A Tutorial

Announcements:

- HW6 due 11/12
- Final Exam 12/12

<u>Recall Def</u>: $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$. Then f is uniformly cts on A if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t. for $x, u \in A$ if $|x-u| < \delta$, then $|f(x) - f(u)| < \varepsilon$.

Nonuniform Cty: TFAE:

1) f is <u>not</u> uniformly cts on A

2) $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$ and $x_\delta, u_\delta \in A$ s.t. $|x_\delta - u_\delta| < \delta$, and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

3) $\exists \varepsilon_0 > 0$ and sequences $(x_n), (u_n)$ in A s.t. $\lim_{n \to \infty} (x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

<u>Thm</u>: Let I be a closed and bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then f is uniformly cts on I.

Q1: Show that $f(x) = \frac{1}{x}$ is not uniformly cts on $(0,\infty)$.

Pf: $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$. Then $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0$.

$|f(x_n) - f(y_n)| = |n - n - 1| \geq 1$.

Q2: Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ satisfy: $\forall \varepsilon > 0$, $\exists g_\varepsilon: A \to \mathbb{R}$ s.t. g_ε is uniformly cts on A and $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Show that f is uniformly cts on A.

Pf: Let $\varepsilon > 0$, $x, y \in A$. $\exists g_{\varepsilon/3}: A \to \mathbb{R}$ s.t.

- $|f(x) - g_{\varepsilon/3}(x)| < \frac{\varepsilon}{3}$
- $|f(y) - g_{\varepsilon/3}(y)| < \frac{\varepsilon}{3}$
- $\exists \delta$ s.t. if $|x-y| < \delta$, $|g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| < \frac{\varepsilon}{3}$.

So if $|x-y| < \delta$, then

$$|f(x) - f(y)| \leq |f(x) - g_{\varepsilon/3}(x)| + |g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| + |g_{\varepsilon/3}(y) - f(y)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Q3: $f: \mathbb{R} \to \mathbb{R}$ is periodic if $\exists p > 0$ s.t. $f(x+p) = f(x)$ for all $x \in \mathbb{R}$.

Prove that if f is continuous and periodic on $\mathbb{R}$, then it is bounded and uniformly cts on $\mathbb{R}$.

Pf: Idea: Since f is periodic, it is completely determined by its values on $[0,p]$.

Boundedness: By cty, $f|_{[0,p]}$ is bounded (Thm 5.3.2), i.e. $\exists M \in \mathbb{R}$, s.t. $|f(a)| \leq M$ for all $a \in [0,p]$.

Now let $b \in \mathbb{R}$. By A.P. $\exists$ largest integer N s.t.

$$Np < b < (N+1)p \iff N < \frac{b}{p} < N+1$$

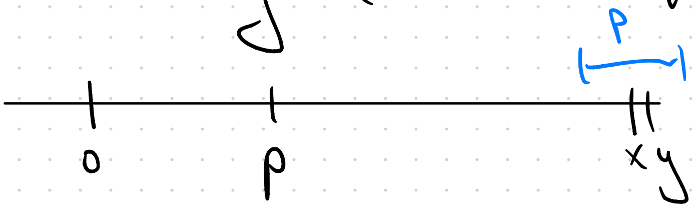
Then $0 < b - Np < Np + p - Np \Rightarrow b \in [0,p)$.

Then by periodicity $|f(b)| = |f(b-Np)| \leq M$.

By cty, f is uniformly cts on $[0,2p]$: $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. if $x, y \in [0,2p]$ with

$|x-y|<\delta_\varepsilon$, then $|f(x)-f(y)|<\varepsilon$. WLOG, take $x<y$.

Now let $x,y\in\mathbb{R}$. And let $|x-y|<\delta=\inf\{\delta_\varepsilon, p, 1\}$.



By A.P., $\exists N\in\mathbb{N}$ s.t.

$$Np<x<y<(N+1)p.$$

$\Rightarrow x-Np,\ y-Np\in[0,2p]$, and

$|x-Np-(y-Np)|=|x-y|<\delta<\delta_\varepsilon$. so,

$|f(x)-f(y)|=|f(x-Np)-f(y-Np)|<\varepsilon$., so f is uniformly cts on $\mathbb{R}$.

$\mathbb{R}$

- sup, inf (equivalent def'ns)
- density of $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$ in $\mathbb{R}$.
- A.P.
- Completeness
- M.I.
- inequalities (AM-GM, triangle, reverse triangle, Cauchy-Schwarz)
- Nested interval thm

Sequences

- limsup, liminf (equivalent def'ns).
  under certain conditions, $\limsup x_n \leq \limsup y_n$.
  $\forall \varepsilon > 0$, infinitely many x_n s.t. $x_n \leq \limsup x_n + \varepsilon$.
- Bolzano-Weierstrass
- Cauchy Sequence
- ε-N def'n of convergence
- Monotone Convergence Thm.
- Divergence

Functions

- cts
- uniform cts
- ε-δ def'n.
- divergence criteria
- cluster pt.
- sequential criteria for convergence
- intermediate value thm
- max-min